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B.Sc. Physics Core Course Semester 2
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PH2CRT02: MECHANICS AND PROPERTIES OF MATTER

(Hydrodynamics)

Part 1: Viscosity

Important Information

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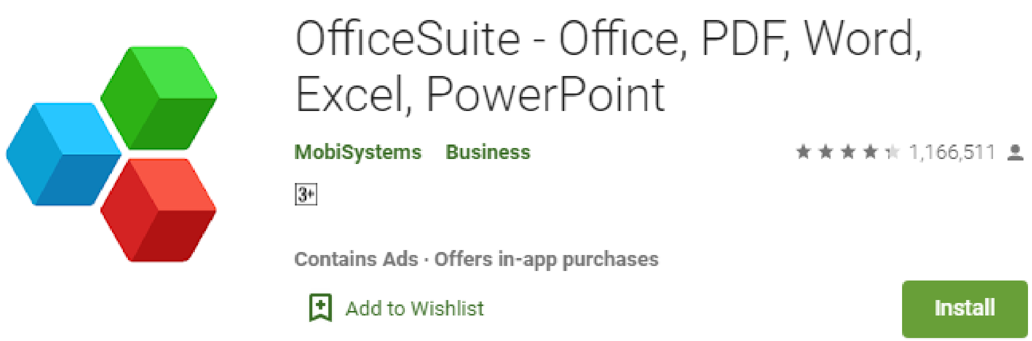

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Hydrodynamics

Liquids and gases can flow and are therefore, called fluids. Although fluids, in general, doesn't have definite shape of its own, liquids assume the shape of its container for a fixed volume, whereas a gas fills the entire volume of its container.

If we apply an external pressure on a liquid, change in its volume is small compared to a gas. We can say that liquids have much lower compressibility compared to gases. A fluid deforms easily if we apply a shear stress (tangential force), which enable them to flow.

Fluid dynamics describes the flow of fluids, both liquids and gases, whereas **hydrodynamics** deals with the liquids in motion. *Viscosity* and *surface tension* are two important properties of a liquid.

Streamline and Turbulent flows

Let us start with a simple daily observation. When you open a water tap slowly, the water flow seems to be very smooth, but if you open the tap more and allow more water to flow, then it loses its smoothness.

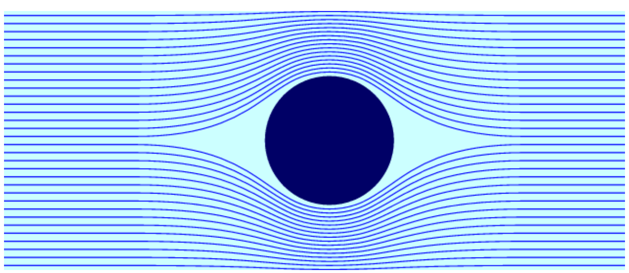
Imagine what is happening to various liquid particles coming one after the other at a particular point within the flow. In the case of a smooth flow the velocity of each passing fluid particle remains constant and the flow is said to be **steady** and the path followed by a particle under a steady flow is called a **streamline**. [Please note that the term "velocity" is used, which means that both magnitude and direction remains same]. The tangent to the streamline at any point gives the *direction of flow* of the liquid at that point. A streamline can be straight or curved path as shown in the figure.

Streamline and Turbulent flows

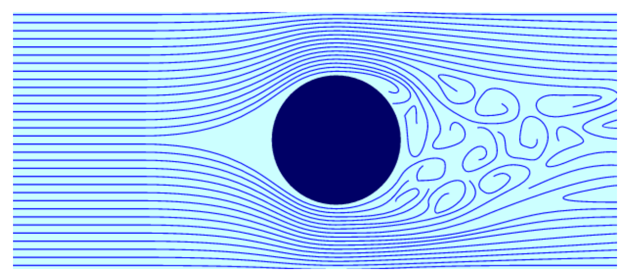
As we have seen in the previous example, the flow is streamline only at lower velocities. When the velocity of flow exceeds a particular velocity, called the *critical velocity*, the flow is no longer streamline but more like a zigzag one and is called **turbulent** flow.

Similar to the turbulent flow of water from a full open water-tap, water flowing through a rocky river is also an example of turbulent flow.

Streamline flow



Turbulent flow



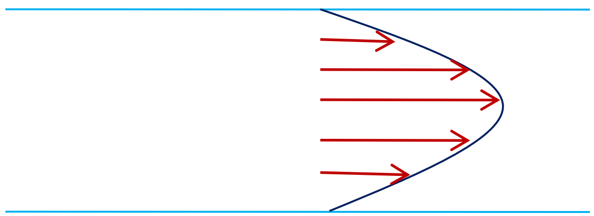
Note that in a streamline flow, different layers flow separately without mixing, whereas in a turbulent flow intermixing of different layers takes place.

Viscosity

Viscosity of a fluid is a measure of its resistance to deformation at a given rate. Honey is more viscous than water.

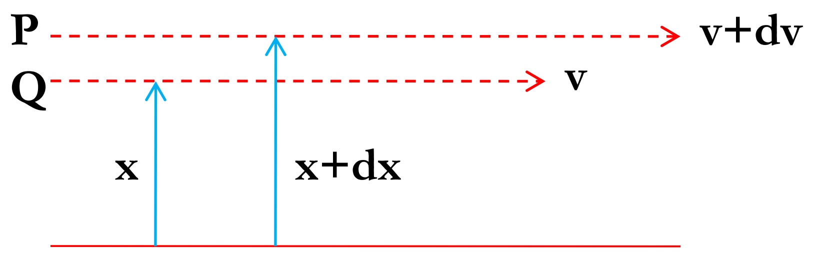
Viscosity of a liquid arises due to the internal frictional force between adjacent layers of liquid that are in relative motion. Consider the case of a laminar flow of a liquid through a tube, it flows with a higher speed near the tube's axis than near its walls.

In fact, the velocity profile across the cross-section of the tube will be parabolic as shown in figure below (we will see it in the next section)



Viscosity

Consider two adjacent layers **P** and **Q** of a streamline flow over a horizontal surface, separated by a distance **dx** and having velocities **v** and (**v+dv**).



Since the velocity of upper layer is slightly greater than the lower layer, a relative motion is set up between the layers P and Q. The upper layer will experience a force dragging it backwards by the lower layer and this force is known as *internal friction* or force of viscosity. The viscous force acts tangentially on the layers.

The property by virtue of which a liquid opposes the relative motion between different layers is known as **viscosity**.

Viscosity

The velocity gradient, between the two layers P and Q is $\frac{dv}{dx}$

It was Newton who showed that the tangential viscous force ***F*** acting on a layer is directly proportional to the area ***A*** of the layer and the velocity gradient.

That is
$$F \propto A\frac{dv}{dx}$$

or
$$F = -\eta A\frac{dv}{dx} \quad \textbf{(Newton's law of viscous flow)} \qquad (1)$$

where **η** is a constant called the **coefficient of viscosity (Nsm^{-2})** of the liquid.

If A=1 and dv/dx = 1, then F= η **.** Thus the coefficient of viscosity of a liquid may be defined as the tangential force acting on unit area of a liquid layer moving with a unit velocity gradient normal to the direction of flow.

Negative sign indicates that viscous force acts opposite to the direction of flow.

Critical Velocity

The velocity at which a streamline flow changes to turbulent flow is known as **critical velocity**. Reynolds proved experimentally that the critical velocity $\boldsymbol{v}_{\mathbf{c}}$ of a liquid depends on the coefficient of viscosity **η**, density **ρ** of the liquid and the radius **r** of the tube through which the liquid is flowing

$$v_c = \frac{K\eta}{\rho r} \qquad (2)$$

Where K is a constant called **Reynold's number**. Typically a value below about 2000 for Reynold's number corresponds to a laminar flow.

From the above equation it is clear that *highly viscous*, *low density* liquid flowing through a narrow tube will follow a streamline profile. Now you know why you use capillary tubes are used for measuring viscosity of liquids in physics labs.

Poiseuille's Equation

Consider a liquid flowing through a horizontal capillary tube under a difference in pressure between the ends of the tube. Poiseuille was able to derive an equation for the amount of liquid flowing per second through the tube, under following assumptions.

(1) The *flow is streamline*, parallel to the axis of the tube

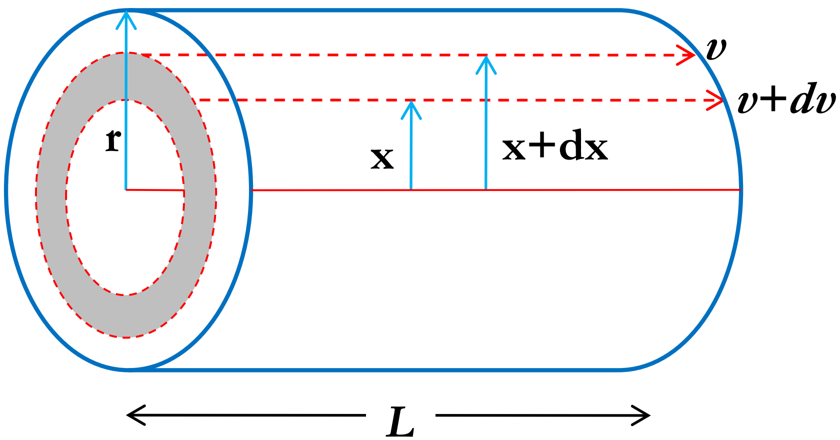
(2) The pressure over any cross-section at right angles to the axis of the tube is constant, that is *no radial flow of liquid*.

(3) The *velocity* of the liquid layer *in contact with the tube is zero* and it *increases gradually towards the axis* of the tube.

(4) Since the tube is placed horizontal, the *gravity does not influence* the flow.

Poiseuille's Equation

Consider a the streamline flow of a liquid of density $\boldsymbol{\rho}$ through a horizontal capillary tube of length $\boldsymbol{L}$ and radius $\boldsymbol{r}$.



As assumed, the velocity of liquid in contact with the walls of the tube is zero and it increases towards the axis. Consider a cylindrical layer of liquid having a radius $\boldsymbol{x}$ and another coaxial layer having radius $\boldsymbol{x+dx}$. Let dv be the velocity difference between the two layers.

Poiseuille's Equation

The velocity gradient between the two layers is $\dfrac{dv}{dx}$

According to Newton's law of viscous force

$$F = -\eta A \frac{dv}{dx}$$

Where ***η*** is the coefficient of viscosity and ***A*** is the surface area of the cylinder of length ***L*** and radius ***x***.

$$A = 2\pi x L$$

$$F = -\eta 2\pi x L \frac{dv}{dx}$$

If P is the pressure difference between the ends of the tube, then the force driving the liquid forward = Pressure × Area = $P.\pi x^2$.

For a steady flow, this driving force is equal to the viscous force

$$P\pi x^2 = -\eta 2\pi x L \frac{dv}{dx}$$

Poiseuille's Equation

Rearranging, we get

$$dv = -\frac{Px}{2\eta L}dx$$

Integrating,

$$\int dv = -\int \frac{Px}{2\eta L}dx$$

$$v = -\frac{P}{2\eta L}\frac{x^2}{2} + C \qquad (3)$$

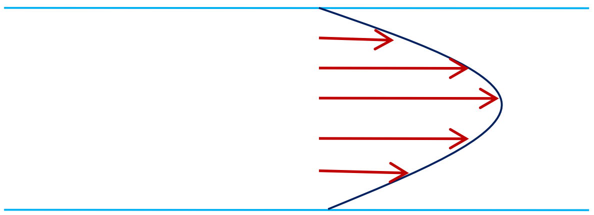
To find the constant of integration, C, at the walls of the tube ***x* = *r*, *v* = *0***.

then $$0 = -\frac{P}{2\eta L}\frac{r^2}{2} + C$$

or $$C = \frac{Pr^2}{4\eta L}$$

Poiseuille's Equation

Substituting for C in equation (3),



$$v = -\frac{Px^2}{4\eta L} + \frac{\Pr^2}{4\eta L}$$

$$v = \frac{P}{4\eta L}\left(r^2 - x^2\right) \qquad (4)$$

Above equation gives the velocity of the liquid layer at a distance ***x*** from the axis of the tube.

If this layer has a finite thickness of *dx*, then the volume of the liquid flowing through this layer per unit time ***dQ*** = *velocity* × *cross-sectional area of the layer.*

$$\boldsymbol{dQ = v \times 2\pi\, x\, dx}$$

$$dQ = \frac{P}{4\eta L}\left(r^2 - x^2\right)2\pi x dx = \frac{\pi P}{2\eta L}\left(r^2 - x^2\right)x dx$$

Poiseuille's Equation

Total volume flowing per second through the tube can be obtained by integrating the above expression over the entire cross-section

$$Q = \int_0^r \frac{\pi P}{2\eta L}\left(r^2 - x^2\right)x\,dx = \int_0^r \frac{\pi P}{2\eta L}\left(r^2 x - x^3\right)dx$$

$$= \frac{\pi P}{2\eta L}\left[r^2\frac{x^2}{2} - \frac{x^4}{4}\right]_0^r = \frac{\pi P}{2\eta L}\left(\frac{r^4}{2} - \frac{r^4}{4}\right)$$

$$Q = \frac{\pi \Pr^4}{8\eta L} \qquad (5)$$

This is the ***Poiseuille's equation for the rate of flow*** of a liquid through a capillary tube. If the liquid is flowing through the capillary tube for a time ***t*** seconds, then the **total volume** is given by

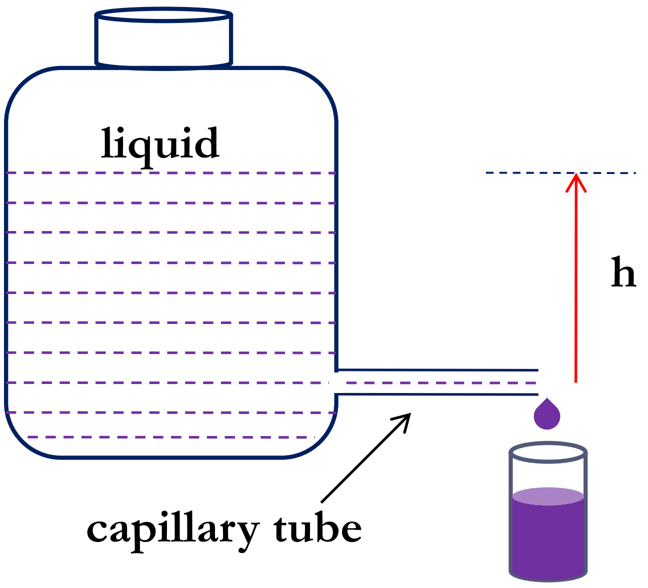
$$V = Qt = \frac{\pi \Pr^4 t}{8\eta L} \quad \text{or} \quad \eta = \frac{\pi P r^4 t}{8VL} \qquad (6)$$

Measurement of Viscosity using Poiseuille's Equation

(1) Constant Pressure Head Method

In the constant pressure head method, we use a large volume container to hold the liquid so that even after the flow of the liquid through the capillary tube for a finite time, the liquid level in the container remains approximately at the same height.

The capillary tube connected at one side near the bottom of the container is placed in the horizontal plane.



Constant Pressure Head Method

Now the liquid is allowed to flow through the capillary tube for a known time $\boldsymbol{t}$ and the volume $\boldsymbol{V}$ thus collected is measured. Height $\boldsymbol{h}$ of the water level from the axis of the capillary tube is measured. The value $\boldsymbol{ht/V}$ is then calculated. The experiment is repeated for different $\boldsymbol{t}$ or $\boldsymbol{h}$ and in each case $\boldsymbol{ht/V}$ is calculated. Length $\boldsymbol{L}$ of the capillary tube is measured using a meter scale and radius $\boldsymbol{r}$ of the capillary tube using a travelling microscope. Using the known value of the density $\boldsymbol{\rho}$ of the liquid, coefficient of viscosity of the liquid is calculated using the formula;

$$\eta = \frac{\pi P r^4 t}{8VL}$$

Here the pressure exerted by the liquid column of height $\boldsymbol{h}$ is $\boldsymbol{P = h\rho g}$

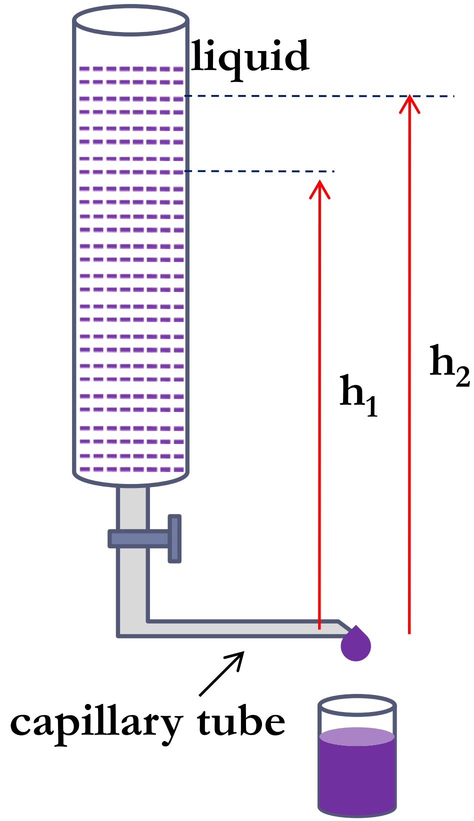
then

$$\eta = \frac{\pi \rho g r^4}{8L} \frac{ht}{V}$$

Measurement of Viscosity using Poiseuille's Equation

(1) Variable Pressure Head Method

In the variable pressure head method, a burette (a long narrow glass tube) is used as the liquid container. Bottom of the burette is connected to a horizontally placed capillary tube. As the liquid flows through the capillary tube, the height of the liquid level in the burette decreases and hence the pressure head changes.



Time ***t*** taken to flow a small volume ***V*** of the liquid (say 1 ml) is noted. Let h_1 and h_2 are the initial and final height of the liquid level in the burette from the axis of the capillary tube.

Variable Pressure Head Method

Average pressure head $\boldsymbol{h = (h_1+h_2)/2}$ is calculated and from that ***ht*** is calculated. The experiment is repeated for different $\boldsymbol{h_1}$ and $\boldsymbol{h_2}$, but same volume ***V*** and in each case ***t*** is measured and the product ***ht*** is calculated as earlier and it will be a constant.

Length ***L*** of the capillary tube is measured using a meter scale and radius ***r*** of the capillary tube using a travelling microscope. Using the known value of the density $\boldsymbol{\rho}$ of the liquid, coefficient of viscosity of the liquid is calculated using the formula;

$$\eta = \frac{\pi P r^4 t}{8VL}$$

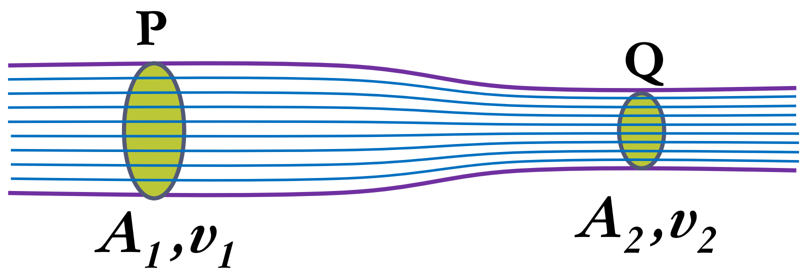
Here the pressure exerted by the liquid column of height ***h*** is $\boldsymbol{P = h\rho g}$

then

$$\eta = \frac{\pi \rho g r^4}{8L} \frac{ht}{V}$$

Equation of Continuity

Consider the streamline flow of a liquid through a non-uniform pipe having different area of cross-section as shown in figure.



Let $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ be the area of cross-section of the pipes at the point **P** and **Q**. Let $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$ be the average velocities at that two points. Assuming that the liquid is incompressible, let $\boldsymbol{\rho}$ be the density of the liquid at P and Q.

Since no liquid is added or lost in between the two ends of the pipe, the amount of liquid flowing through any cross-section of the pipe in a given time is same. Hence, mass of the liquid flowing per second at both point P and Q should be the same.

Equation of Continuity

That is $\boldsymbol{A_1 v_1 \rho = A_2 v_2 \rho}$

or $\boldsymbol{A_1 v_1 = A_2 v_2}$

or $\boldsymbol{Av = constant}$ (7)

This is the *equation of continuity for the streamline flow of an incompressible liquid.*

Av gives flow rate and it remains constant throughout the pipe of flow. Thus, at narrower portions where the streamlines are closely spaced, velocity increases and its vice versa.

Energy possessed by a liquid

A liquid undergoing streamline flow possesses three forms of energy, namely

(1) Kinetic Energy

(2) Potential Energy

(3) Pressure Energy

(1) Kinetic Energy

Consider a liquid of mass $\boldsymbol{m}$ and density $\boldsymbol{\rho}$ is flowing with a velocity $\boldsymbol{v}$, then the

Kinetic energy possessed by the liquid $= \frac{1}{2}mv^2$

Kinetic energy per unit mass $= \frac{1}{2}v^2$

Kinetic energy per unit volume $= \frac{1}{2}\frac{mv^2}{V} = \frac{1}{2}\frac{(V\rho)v^2}{V} = \frac{1}{2}\rho v^2$

Energy possessed by a liquid

(2) Potential Energy

Consider a liquid of mass $\boldsymbol{m}$ is situated at a mean height of $\boldsymbol{h}$, above some reference level then the

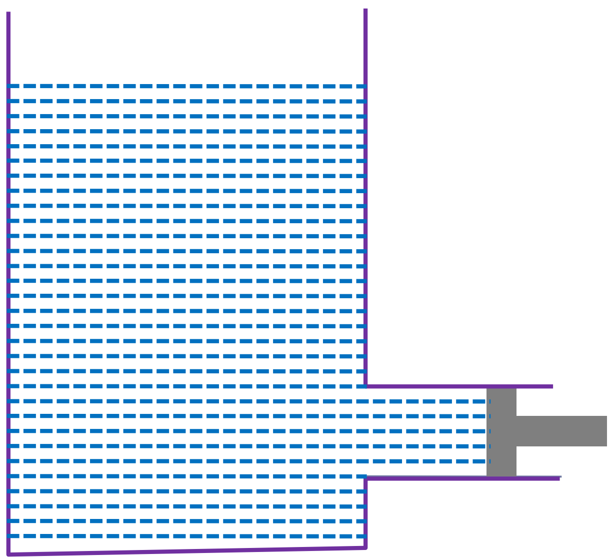
Potential energy possessed by the liquid = $\boldsymbol{mgh}$

Potential energy per unit mass = $\boldsymbol{gh}$

Potential energy per unit volume $= \dfrac{mgh}{V} = \dfrac{V\rho gh}{V} = \rho gh$

Energy possessed by a liquid

(3) Pressure Energy



Consider a beaker fitted with a frictionless piston at the bottom side tube attached to the beaker as shown in figure.

Let $\boldsymbol{\rho}$ be the density of the liquid filled.

Let $\boldsymbol{a}$ be the area of cross-section of the piston.

Let $\boldsymbol{p}$ be the pressure experienced by the piston due to the finite water column in the beaker.

Energy possessed by a liquid

If we push the piston through a distance $\boldsymbol{x}$ towards the beaker, the volume of the liquid pushed back into the beaker, $\mathbf{V} = \boldsymbol{ax}$

Mass of the liquid pushed back, $\boldsymbol{m} = \boldsymbol{V\rho}$

Force applied on the piston = pressure × area = $\boldsymbol{pa}$

Work done on the piston = Force × displacement = $\boldsymbol{pax} = \boldsymbol{pV}$

This work done is the pressure energy of a mass $\boldsymbol{m} = \boldsymbol{V\rho}$ of the liquid.

Hence, the pressure energy per unit mass $= \dfrac{pV}{V\rho} = \dfrac{p}{\rho}$

Pressure energy per unit volume $= \dfrac{pV}{V} = p$

Bernoulli's Theorem-Statement

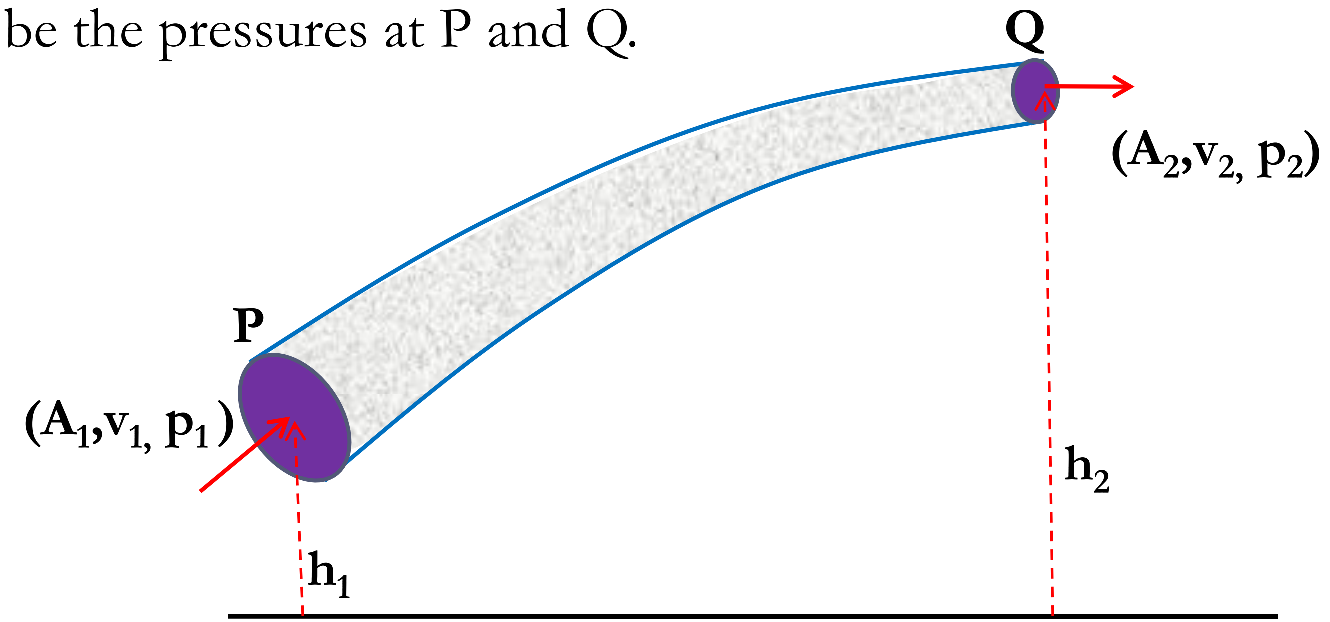
Bernoulli's theorem states that as we move along a streamline the total energy or the sum of the pressure energy, the kinetic energy per unit volume and the potential energy per unit volume remains a constant.

that is $$\frac{P}{\rho}+\frac{v^2}{2}+gh=constant$$

Since Bernoulli's theorem is a conservation of energy theorem, *it is applicable only for incompressible non-viscous liquids undergoing streamline flow*. If the liquid is viscous in nature, then there will be energy loss due to higher frictional force. If the flow is turbulent, then the pressure and velocity of the liquid are randomly fluctuating and the above theorem is no longer valid.

Bernoulli's Theorem-Proof

Consider a pipe positioned at varying heights as shown in figure. Consider a fluid moving through the pipe of varying cross-sectional area $\boldsymbol{A_1}$ and $\boldsymbol{A_2}$ at P and Q at heights $\boldsymbol{h_1}$ and $\boldsymbol{h_2}$ from a reference point. If an incompressible fluid of density $\boldsymbol{\rho}$ is flowing through the pipe in a streamline, its velocity must change as a consequence of equation of continuity and let it be $\boldsymbol{v_1}$ and $\boldsymbol{v_2}$. Let $\boldsymbol{p_1}$ and $\boldsymbol{p_2}$ be the pressures at P and Q.



Bernoulli's Theorem-Proof

In a small time interval $\boldsymbol{\delta t}$, the liquid at point P moves a distance $\boldsymbol{v_1 \delta t}$

Force acting at the point P = $\mathbf{A_1 p_1}$

Work done in moving the liquid through a distance $\boldsymbol{v_1 \delta t}$ at point A is

$$W_1 = \text{Force} \times \text{distance}$$

$$= \mathbf{A_1 p_1 \, v_1 \delta t}$$

Similarly, work done in moving the liquid through a distance $\boldsymbol{v_2 \delta t}$ at point B is

$$W_2 = \mathbf{A_2 p_2 \, v_2 \delta t}$$

But, according to equation of continuity, $\mathbf{A_1 v_1 = A_2 v_2}$

Therefore $\quad W_2 = \mathbf{A_1 p_2 \, v_1 \delta t}$

Net work done by the pressure $W = \mathbf{W_1 - W_2 = (p_1 - p_2) A_1 v_1}$ (8)

This work done is partially utilized for lift the liquid from height $\mathbf{h_1}$ to height $\mathbf{h_2}$ and the remaining is used to provide the flow velocity.

Bernoulli's Theorem-Proof

The work done in lifting the liquid from h_1 to h_2 = $\mathbf{mg(h_2\text{-}h_1)}$

This work is converted to the potential energy of the liquid.

Now, the gain in kinetic energy due to change in velocity$= \frac{1}{2}m\left(v_2^2 - v_1^2\right)$

Therefore, the total change in energy of the mass m in a time $\boldsymbol{\delta t}$ is

$$= mg\left(h_2 - h_1\right) + \frac{1}{2}m\left(v_2^2 - v_1^2\right) \tag{9}$$

Since, this energy is provided by the work done by the pressure, equate (8) and (9), we get

$$(p_1 - p_2)A_1v_1 = mg\left(h_2 - h_1\right) + \frac{1}{2}m\left(v_2^2 - v_1^2\right) \tag{10}$$

Bernoulli's Theorem-Proof

But, mass **m** of the liquid having density $\boldsymbol{\rho}$ flowing through an area of cross-section $\mathbf{A_1}$ with a velocity $\mathbf{v_1}$ is given by **m** = volume × density = $\mathbf{A_1}\ \mathbf{v_1}\ \boldsymbol{\rho}$

Substitute for m in (10), we get,

$$(\mathrm{p}_1 - \mathrm{p}_2)\mathrm{A}_1\mathrm{v}_1 = \mathrm{A}_1\mathrm{v}_1\rho\, g\left(h_2 - h_1\right) + \frac{1}{2}\mathrm{A}_1\mathrm{v}_1\rho\left(v_2^2 - v_1^2\right)$$

cancelling $\mathbf{A_1}\ \mathbf{v_1}$

$$(\mathrm{p}_1 - \mathrm{p}_2) = \rho\, g\left(h_2 - h_1\right) + \frac{1}{2}\rho\left(v_2^2 - v_1^2\right)$$

rearranging

$$\mathrm{p}_1 + \frac{\rho}{2}v_1^2 + \rho g h_1 = \mathrm{p}_2 + \frac{\rho}{2}v_2^2 + \rho g h_2$$

dividing throughout by $\boldsymbol{\rho}$ we get,

$$\frac{\mathrm{p}_1}{\rho} + \frac{v_1^2}{2} + g h_1 = \frac{\mathrm{p}_2}{\rho} + \frac{v_2^2}{2} + g h_2$$

Bernoulli's Theorem-Proof

or we can write

$$\underbrace{\frac{p}{\rho}}_{\text{Pressure energy per unit mass}} + \underbrace{\frac{v^2}{2}}_{\text{Kinetic energy per unit mass}} + \underbrace{gh}_{\text{Potential energy per unit mass}} = constant \qquad (11)$$

Hence, *Bernoulli's theorem is proved.*

Dividing (11) throughout by ***g***, we get $\frac{p}{\rho g} + \frac{v^2}{2g} + h = constant$

Since each term has the dimension of length, each term in the above equation is called ***pressure head***, ***velocity head*** and ***gravity head*** respectively.